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An analytical framework for hybrid spiral-ellipse trajectories under atmospheric drag

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Abstract

This paper presents a framework of a comprehensive analytical solution for the trajectory of a space object in an atmosphere. Analyzing this dynamical system needs numerical simulations or approximated solutions due to lack of a comprehensive analytical solution. This paper demonstrates that this trajectory is characterized by an exponential plus oscillational functions. Consequently, this paper establishes describing the trajectoy in an atmosphere through analytical solutions. Departing from traditional inertia-resistance models, this paper proposes the framework with an empirical law governing trajectory in an atmosphere. The geometry and equation of the resulting solution represent a superposition of spiral and elliptical trajectory, corresponding to the observed exponential plus oscillational behavior. This solution contains the conic section equation, enabling adaptation to various scales of atmospheric drag and eccentricity. Demonstration against numerical simulations using the traditional model reveals approximately 1% error in ellipsoidal heights during reentry when employing nonlinear curve fitting with the proposed solution.
Keywords: Atmospheric drag, Reentry, Analytical solution, Kepler's law, Spiral, Ellipse

Nomenclature

A	area, m^2
a	semi-major axis, km
a_r	radial acceleration, km/s^2
a_θ	transverse acceleration, km/s^2
B	elliptic parameter, km^{-1}
Bi	secondary Airy function
b	$a\sqrt{1-e^2}$; semi-minor axis
C_D	drag coefficient
e	eccentricity
F_D	drag force, km/s^2
f	true anomaly, rad
$g(\theta)$	abbreviation of spiral element
h	angular momentum, km^2/s
m	mass, kg
r	radius, km
S	swept area, km^2
T	period, s
t	time, s
u	$\omega + f$; argument of true latitude, rad
v	velocity, km/s
α_1	spiral parameter, km^2/s
α_2	spiral parameter
θ	cumulative polar angle, rad
θ_e	polar angle at reentry, rad
$\mu_\oplus$	geocentric gravitational constant, km^3/s^2
$\mu_\odot$	heliocentric gravitational constant, km^3/s^2
ρ	atmospheric density, kg/m^3
σ	$1/r$; reciprocal radius, km^{-1}
ω	argument of perigee, rad
$(\dot{\ })$	time differential

Subscript	
0	initial value

1. Introduction

Analytical solutions provide conceptual insights to kinematics, shortest computational costs, flexible applicability, and standards for perturbation theories. There are various dynamical systems that have not had analytical solutions yet, and the trajectory of a space object under an atmospheric drag is one of them. This trajectory obviously contains the elements of ellipse and spiral; however, no framework for analytically describing this dynamical system without any approximation exists.

Numerical simulation is a common method for the analyzing trajectory in an atmosphere due to the complexity of integrating the traditional drag model, which is the inertia resistance. For that, various analytical solutions during atmospheric decay and reentry are proposed in the literature. In low Earth orbits, atmospheric drag is regarded as a perturbation effect. The change of the orbital elements is described as the Bessel functions with simplifications such as averaging, exponential density, and low or high eccentricity [1]. For reentry, approximations such as near circular orbits [2] and constant flight-path angle [3] are used. These analytical solutions are useful for analyzing kinematic features in each case. Nevertheless, these formulations employ some kinds of approximation, then equations and theories must be chosen according to the orbital environments.

This paper establishes a comprehensive framework to analytically describe the spiral-ellipse trajectory of a space object in an atmosphere. Specifically, this paper proposes an integrable drag model. Phenomenons and expectations are often modeled with rough main frameworks and strict fine-tunings. This structure is found in various theoretical layers such as the Newtonian mechanics and the relativity in motion, the two-body problem and perturbations in astrodynamics, the square of velocity and atmospheric density model in the inertia resistance, etc. The main frameworks are often easy to understand and intuitive for humans. The astrodynamics in an atmosphere is mainly described by the gravity and the inertia resistance. However, this is combination of the two frameworks, in other words, a single framework can be rebuilded.

This paper is composed of the following sections. Section 2 shows a trajectory in the atmosphere with a traditional numerical simulation to discover a potential for the analytical solution. Then, a modeling method for the analytical solution is introduced with the inspired example of the Kepler's laws and Newton's universal gravity. The dynamical systems of the two-body problem and one with atmospheric drag are formulated to support the modeling. Section 3 models the trajectory in the atmosphere by proposing empirical laws. Then, the analytical solution is obtained and verified. Section 4 concludes the features of the dynamical system in an atmosphere in the modeling.

2. Preliminaries

2.1 Preliminary analysis

This section presents numerical simulations of orbital decay and reentry with the traditional force model and explains the potential to analytically describe the trajectory. Figures 1 and 2 show an example of trajectory in the atmosphere simulated with the traditional inertia resistance model and NRLMSISE-00 model, one of atmospheric models. The simulation ends at the altitude of 70 km. The geocentric distance oscillates and finally plunges. The angular momentum decreases in a stepwise pattern.

This study supposes that the appearance of the geocentric distance at the plunge is an feature of importance. This feature of oscillation and plunge is rare but can be observed in a function. That is the secondary Airy function which is defined as

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3}+xt\right)+\sin\left(\frac{t^3}{3}+xt\right)\right] dt \quad (1)$$

The appearance of oscillation and plunge corresponds the equation of exponent plus sine above. Figure 3 shows the appearance of $-\mathrm{Bi}$ that is similar to that of the geocentric distance. This means a potential to analytically describe the trajectory in an atmosphere. Thus, the analytical solution contains the sum of oscillational and exponential (or this kind of monotonic) functions.

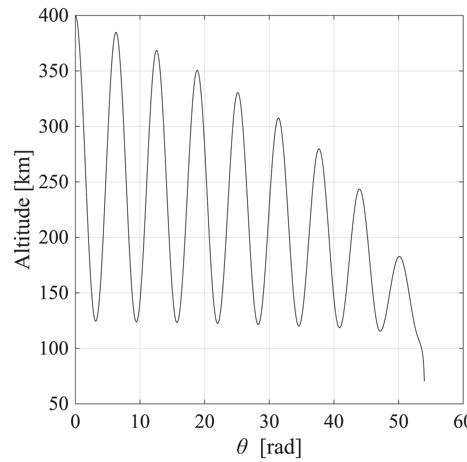

Fig. 1: History of altitude (geocentric distance)

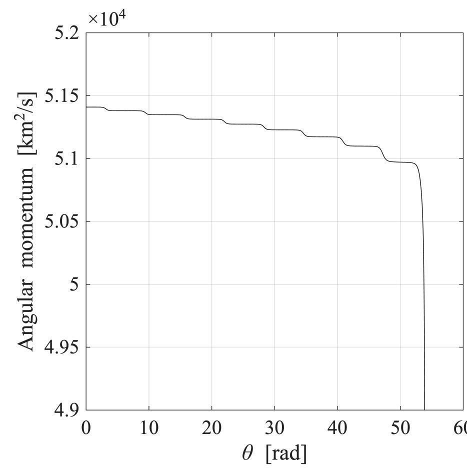

Fig. 2: History of angular momentum

2.2 Kinematic interpretation of Kepler's law

This section demonstrates how Kepler's law determines the force model of universal gravity, and this paper utilizes the process for modeling in Sec. 3. The following in this section mainly refers to [4].

Kepler's laws consist of the following three:

1. First Law (Law of Ellipses): Planets orbit the Sun in elliptical orbits with the Sun at one focus.
2. Second Law (Law of Equal Areas): A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. Third Law (Law of Periods): The square of a planet's orbital period is directly proportional to the cube of the semi-major axis of its orbit.

These laws requests acceleration expressions as follows.

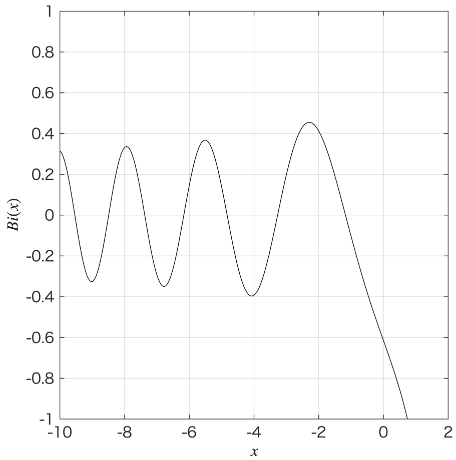

Fig. 3: *Bi* function (sign inversion)

From the second law,

$$\frac{\Delta S}{\Delta t} = \frac{\pi ab}{T} \tag{2}$$
$$= \frac{dS}{dt}$$
$$= \frac{1}{2} r^2 \dot{\theta}$$
$$= h/2 \text{ (const.)} \tag{3}$$

Noting that h here is not deliberately called angular momentum as the reserved quantity, but it is just a constant requested from the Kepler's second law. Since differentiating the constant, the acceleration in the transverse direction becomes

$$\begin{aligned} a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} \\ &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \\ &= 0 \end{aligned} \tag{4}$$

The acceleration in the radial direction can be converted into

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= \frac{d\theta}{dt}\frac{d}{d\theta}\dot{r} - \frac{r^4\dot{\theta}^2}{r^3} \\ &= \dot{\theta}\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right] - \frac{h^2}{r^3} \\ &= -\left(\frac{h}{r}\right)^2 \left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right] \end{aligned} \tag{5}$$

The first law requests conic section and its reciprocal is

$$\frac{1}{r} = \frac{1 + e\cos\theta}{a(1-e^2)} \tag{6}$$

Differentiating the equation above yields

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{e\cos\theta}{a(1-e^2)} \tag{7}$$

Using $h/2 = \pi ab/T = \pi a^2\sqrt{1-e^2}/T$ and substituting Eqs. (6) and (7) into Eq. (5) yields

$$a_r = -\left(\frac{h}{r}\right)^2 \frac{1}{a(1-e^2)} = -4\pi^2\left(\frac{a^3}{T^2}\right)\frac{1}{r^2} \tag{8}$$

From the third law, a^3/T^2 is constant, resulting in

$$a_r = -\frac{\mu_\odot}{r^2} \tag{9}$$

where $\mu_\odot$ is a constant and does not depend on planets orbiting the Sun.

The expression of Newton's universal gravity can be derived from Kepler's laws with kinematic interpretation and calculus. This process can be regarded as the approximately same way of Newton at that time. He did not use the word, calculus, but use the think of limit with geometric explanation. Whether the process equals to Newton's way is not a problem, rather, the important thing is that, if calculus existed, one could create the force model of the universal gravity from the Kepler's laws. This force model can be made from the empirical laws, but not from the dynamics. On the other hand, models such as the inertia resistance is dynamics-based from Newton's law of drag. It is reasonable to allow for another types of models such as atmospheric drag model made from empirical laws.

2.3 Two-body problem

This section describes the two-body problem to support the formulation in the following section. The equations of motion in the two-body problem are given as

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu_\oplus}{r^2} \tag{10}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{11}$$

Eq. (11) can be converted into

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{12}$$

and this is known as the angular momentum of the reserved quantities, integrated as

$$r^2\dot{\theta} = h \tag{13}$$

Within the two-body problem, the angular momentum holds constant. The variable transformation from r to

σ is employed and Eq. (10) can be solved as a function of θ. Differentiating σ by θ yields

$$\begin{aligned}\frac{d\sigma}{d\theta} &= \frac{dt}{d\theta}\frac{d\sigma}{dt} \\ &= -\frac{\dot{r}}{r^2\dot{\theta}} \qquad (14)\\ \frac{d^2\sigma}{d\theta^2} &= \frac{dt}{d\theta}\frac{d}{dt}\left(\frac{d\sigma}{d\theta}\right) \\ &= -\frac{\ddot{r}r^2\dot{\theta} - 2r\dot{r}^2\dot{\theta} - r^2\dot{r}\ddot{\theta}}{r^4\dot{\theta}^3} \\ &= -\frac{1}{r^2\dot{\theta}^2}\left[\ddot{r} - \frac{\dot{r}}{r\dot{\theta}}\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right] \qquad (15)\\ &= -\frac{\ddot{r}}{r^2\dot{\theta}^2} \qquad (16)\end{aligned}$$

Substituting Eq. (16) into Eq. (10) yields

$$\begin{aligned}\frac{d^2\sigma}{d\theta^2} + \sigma &= \frac{\mu_\oplus}{r^4\dot{\theta}^2} \\ &= \frac{\mu_\oplus}{h^2} \qquad (17)\end{aligned}$$

Equation (17) can be integrated to

$$\sigma = \frac{\mu_\oplus}{h^2} + B\cos(\theta + \theta_0) \qquad (18)$$

Then, the solution, known as the conic section, is obtained as

$$\begin{aligned}r &= \frac{1}{\sigma} \\ &= \frac{h^2/\mu_\oplus}{1 + (Bh^2/\mu_\oplus)\cos(\theta + \theta_0)} \qquad (19)\end{aligned}$$

According to variables when polar coordinate axes selected at initial time, a pair of $(\theta = 0, \ldots, 2\pi, \theta_0)$ is denoted as (f, f_0) or (u, u_0). The former is still a relative expression between initial and perigee axes. For the latter, by considering node vector that serve as standards to define absolute position of the ellipse in the inertia frame, ω is needed finally.

2.4 Aerodynamics force

This section formulates the equations of motion with aerodynamic forces to support the modeling in Sec. 3. The equations of motion in an atmosphere are given by

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu_\oplus}{r^2} + F_D\frac{\dot{r}}{v} \qquad (20)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = F_D\frac{r\dot{\theta}}{v} \qquad (21)$$

In low Earth orbit, inertial resistance of atmospheric drag is usually employed as

$$F_D = -\frac{1}{2}\rho\frac{A}{m}C_D v^2 \qquad (22)$$

Substituting Eq. (21) into Eq. (20) becomes

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu_\oplus}{r^2} + \frac{\dot{r}}{r\dot{\theta}}(2\dot{r}\dot{\theta} + r\ddot{\theta}) \qquad (23)$$

Similarly, Eq. (23) is converted using Eq. (15) to

$$\frac{d^2\sigma}{d\theta^2} + \sigma = \frac{\mu_\oplus}{h^2} \qquad (24)$$

Interestingly, Eq. (24) completely equals to Eq. (17) even with the drag force. It should be noted that $h = r^2\dot{\theta}$ decreases by the drag force and this is why the equation is difficult to integrate. In other words, Eq. (17) implies that h must be modeled so that the square of its reciprocal is integrable in θ to analytically describe the solution. This formulation is valid as long as F_D is in the direction of velocity, thus other forces such as the rotation of atmosphere should be treated separately as perturbations.

3. Modeling and Verification

3.1 Modeling

Reflecting the results of Sec. 2, this paper proposes a new law of orbit decay, "angular momentum is composed of logistic functions." Basically, each logistic function corresponds to a single revolution, excepting skip reentries. For many cases, the solution of a single revolution is enough to use. Moreover, reentry condition makes angular momentum zero as $h \to 0$ for $\theta \to \infty$, so that integration constant can be set to zero as

$$h = \frac{\alpha_1}{1 + e^{-\alpha_2(\theta - \theta_e)}} \qquad (25)$$

Substituting Eq. (25) into Eq. (24), the radial ODE becomes

$$\begin{aligned}\frac{d^2\sigma}{d\theta^2} + \sigma &= \frac{\mu_\oplus}{h^2} \\ &= \frac{\mu_\oplus}{\alpha_1^2}[1 + 2e^{-\alpha_2(\theta-\theta_e)} + e^{-2\alpha_2(\theta-\theta_e)}] \qquad (26)\end{aligned}$$

The equations of motion with the proposed model can be integrated in terms of a polar angle without any approximation. The solution is composed of the general solution and the specific solution as

$$\sigma = \frac{\mu_\oplus}{\alpha_1^2}[1 + g(\theta)] + B\cos(\theta + \theta_0) \qquad (27)$$

where

$$g(\theta) = \frac{2}{\alpha_2^2 + 1}e^{-\alpha_2(\theta-\theta_e)} + \frac{1}{4\alpha_2^2 + 1}e^{-2\alpha_2(\theta-\theta_e)} \qquad (28)$$

From the definition of σ,

$$r = \frac{\alpha_1^2/\mu_\oplus}{1 + g(\theta) + B\alpha_1^2/\mu_\oplus \cos(\theta + \theta_0)} \quad (29)$$

resulting in the geocentric distance as the function of the cumulating angle. θ_e defines the absolute position at reenter indirectly connecting to the inertia frame, which has the same role of ω. Strictly, θ is a cumulative angle because spiral continues infinitely, but almost reentry trajectories end within 2π. When considering orbits that not reenter, $\theta_e \to \infty$, $g(\theta) \to 0$, and Eq. (29) completely corresponds to the conic section. The geometry and equation of the solution are the superposition of spiral and ellipse, respectively corresponding to the exponential and oscillational functions as expected. The equation of the solution is a generalized form of the conic section and thus can be adapted to various scales of atmospheric drag and eccentricities.

There would be room for considering concrete expression of the logistic function if more accuracy is needed. Generalizing the logistic function would be effective such as

$$h = \frac{\alpha_1}{[1 + e^{-\alpha_2(\theta - \theta_e)}]^{1/2}} \quad (30)$$

which simplifies $g(\theta)$ into a single exponential term. As long as the square of the reciprocal can be integrable, generalized logistic function is suitable for this modeling.

3.2 Verification and systematizing

This section verifies the solution generated from the modeling by the nonlinear curve fitting. Figure 4 demonstrates the accuracy of Eq. (29). The error in ellipsoidal heights during reentry holds approximately 1% between numerical simulations using the traditional model and nonlinear curve fitting to them by the proposed solution.

This results can systematize the first and second laws for the trajectory in an atmosphere as follows.

1 First Law (Law of Spiral Ellipse Sum): Space objects in an atmosphere orbit in spiral-ellipse trajectories whose reciprocal radiuses are exponential (monotonic) plus oscillational functions.

2 Second Law (Law of Stepped Angular Momentum): Angular momentum is composed of sum of logistic functions.

These laws have extracted from numerical simulations, instead of observed values that Kepler used, with the traditional drag model that has been verified for a long period of time. Thus, the reliability of the laws equals to that of the traditional model and the laws are not properties that changes with slight errors of atmospheric models.

As well as Sec. 2.2, accelerations can be requested from the first and second laws above. Thus, the expression of forces are derived by setting Eqs. (29) (or Eq. (27)) and (25) a start.

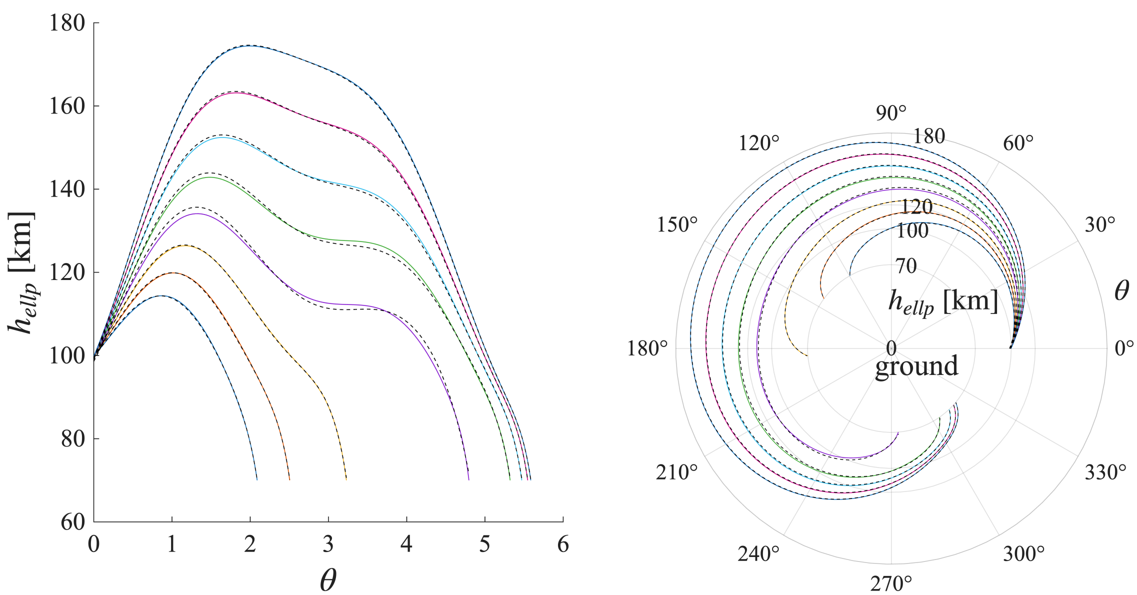

Fig. 4: Curve fitting of reentry trajectories

Using Eq. (15),

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= -r^2\dot{\theta}^2\frac{d^2\sigma}{d\theta^2} + \frac{\dot{r}}{r\dot{\theta}}\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\theta}^2 \\ &= -r^2\dot{\theta}^2\left(\frac{d^2\sigma}{d\theta^2} + \sigma\right) + \frac{\dot{r}}{r\dot{\theta}}\frac{\dot{\theta}}{r}\frac{dh}{d\theta} \end{aligned} \tag{31}$$

From the first law, differentiating Eq. (27) yields

$$\frac{d^2\sigma}{d\theta^2} + \sigma = \frac{\mu_\oplus}{\alpha_1^2}[1 + 2e^{-\alpha_2(\theta-\theta_e)} + e^{-2\alpha_2(\theta-\theta_e)}]$$

From the second law,

$$\frac{d^2\sigma}{d\theta^2} + \sigma = \frac{\mu_\oplus}{h^2} \tag{32}$$

Then, substituting the equation above into Eq. (31) as

$$\begin{aligned} a_r &= -r^2\dot{\theta}^2\frac{\mu_\oplus}{h^2} + \frac{1}{r^2}\frac{dh}{d\theta}\dot{r} \\ &= -\frac{\mu_\oplus}{r^2} + \frac{v}{r^2}\frac{dh}{d\theta}\frac{\dot{r}}{v} \end{aligned} \tag{33}$$

Thus, a_r is divided into the gravity and the other. On the other hand, from the second law,

$$\begin{aligned} a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} \\ &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \\ &= \frac{\dot{\theta}}{r}\frac{dh}{d\theta} \\ &= \frac{v}{r^2}\frac{dh}{d\theta}\frac{r\dot{\theta}}{v} \end{aligned} \tag{34}$$

Consequently, the other acceleration have components of velocity vector, $(\dot{r}, r\dot{\theta})$, in each direction. Thus, this acceleration is in the direction of velocity. The first and second laws determined the direction of acceleration (defined as force later) as well as the Kepler's first and second laws. From the second law, differentiating h in θ as

$$\frac{dh}{d\theta} = \frac{\alpha_2}{\alpha_1}(\alpha_1 - h)h \tag{35}$$

At this stage,

$$F_D = \frac{v}{r^2}\frac{dh}{d\theta} \tag{36}$$

$$= \frac{\alpha_2}{\alpha_1}(\alpha_1 - h)\frac{vh}{r^2} \tag{37}$$

Since there are force models made of r or v, it is reasonable that the model made of h exists. Noting that Eq. (37) is temporary like Eq. (8) and the strict expression of F_d will be determined after discovering the third law.

4. Conclusions

This paper has proposed a framework which reconstructs the atmospheric drag force to analytically describe trajectories in an atmosphere. This paper has featured the system as reciprocal radius as the exponential plus oscillational functions and the angular momentum of logistic function whose square of its reciprocal is integrable. These features are reflected in modeling that is inspired by Newton's universal gravity model with the kinematic interpretation of Kepler's laws. This results in the solution of the reentry trajectory as a conic section with the element of spiral added, therefore it can be used for a variety of eccentricities or drag scales. Future works will explore third laws to constraint the independent parameters of the solution.

Acknowledgment

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